Appl. Math. Lett. Vol. 3, No. 3, pp. 103-105, 1990
Printed in Great Britain.

0893-9659/90 \$3.00 + 0.00

Uniqueness Result for Inverse Problem of Geophysics II

A.G. RAMM AND G.Q. XIE

Mathematics Department, Kansas State University

(Received February 1990)

Abstract. It is proved that an analytic in a bounded region inhomogeneity can be uniquely recovered from the surface measurements of the acoustic field for a fixed position of the source and all frequencies.

Introduction.

Let

$$\ell_v u := \left[\nabla^2 + k^2 + k^2 v(x)\right] u(x,k) = -\delta(x) \quad \text{in } \mathbb{R}^3, \quad x = (x_1, x_2, x_3) \tag{1}$$

$$u \quad \text{satisfies the radiation condition,} \quad k = \text{const} > 0, \tag{2}$$

$v = 0$ outside $\mathcal{D}$, a bounded domain with a C^∞ boundary, $\mathcal{D} \subset \mathbb{R}^3_-$, $\mathbb{R}^3_- = \{x \mid x \in \mathbb{R}^3, x_3 < 0\}$. Suppose that

$$v(x) \quad \text{is analytic in } \overline{\mathcal{D}}. \tag{3}$$

Actually any condition on v which implies that $v(x)$ is uniquely determined in $\mathcal{D}$ if $D_N^m v$, $m = 0,1,2,\ldots$, are known on Γ will be sufficient. Here $D_N^m v$ is m-th normal derivative of v at the boundary from $\mathcal{D}$. Analyticity in $\overline{\mathcal{D}}$ means that $v(x)$ admits an extension as an analytic function to a domain $\mathcal{D}_1$ which contains $\mathcal{D}$ as a strictly inner subdomain, the overbar stands for completion. The solution to problem (1)-(2) is the acoustic pressure generated by a point source situated at the origin in the homogeneous medium with local inhomogeneity $\mathcal{D}$, $1 + v(x) = c^{-2}(x)$, where $c(x)$ is the wave velocity.

The inverse problem (IP) is: *given $u(x,k)$ for all $k > 0$ and all $x \in \Omega$ where Ω is an arbitrary small open set on the plane $P := \{x \mid x_3 = 0\}$, find $v(x)$.*

The result of this paper is the uniquenes theorem:

THEOREM 1. *The IP has at most one solution if (3) holds.*

In the literature there is one result on the uniqueness of the solution to IP: it is proved in [1] that uniqueness holds if $\frac{\partial v}{\partial x_j} = 0$ in $\mathcal{D}$ for some j, $1 \le j \le 3$, and $v \in H^1(\mathcal{D})$, where $H^\ell(\mathcal{D})$ is the Sobolev space. In [2] it is proved that if the data $u(x,k)$, $x \in \Omega \subset P$ are known, then the data $u(x,k)$, $x \in P$ are uniquely determined. This allows one to assume that the data are given on all of P. In [3] uniqueness theorems are proved for the data given on P for low frequencies and all positions of the source and receiver. In [4]-[6] inverse problems with fixed frequency data are studied. In the next section we prove Theorem 1.

AGR thanks ONR and NSF for support.

PROOF:

The proof follows the ideas of [1]. The new ingredient is an orthogonality relation for $D^\alpha p(x)$, $p(x) := v_2(x) - v_1(x)$, for any multiindex α. This relation implies that $D_N^m p(x) = 0$ on Γ for all $m = 0, 1, 2 \dots$. This and (3) imply that $p(x) = 0$. Here $v_1(x)$ and $v_2(x)$ are two functions which produce the same surface data $u(x,k)$ for all $k > 0$ and all $x \in P$.

The two basic steps of the proof are:

1) derivation of the orthogonality condition

$$\int_{\mathcal{D}_1} \phi(x) \left[D^\alpha(pu_2) - D^\alpha(v_1 w) + v_1 D^\alpha w\right] dx = 0 \quad \forall \phi \in N_1(\ell_{v_1}) \tag{4}$$

where $N_1(\ell_{v_1}) := \{\phi : \ell_{v_1}\phi = 0 \quad \text{in} \quad \mathcal{D}_1\}$, and $\mathcal{D}_1 \supset \mathcal{D}$ is a domain to which $v_1(x)$ admits an analytic extension, and $w = u_1 - u_2$,

and

2) derivation of the equations

$$D_N^m p = 0 \quad \text{on } \Gamma, \quad m = 0, 1, 2 \dots. \tag{5}$$

Step 1. Assume that v_1 and v_2 produce the same surface data $u(x,k)$, $x \in P$, $k > 0$. Let $w := u_1 - u_2$. Apply the operator D^α to equations (1) with $v = v_1$ and $v = v_2$ and subtract the resulting equations, $D^\alpha := \frac{\partial^{\alpha_1+\alpha_2+\alpha_3}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}$, differentiation is understood in the sense of the theory of distributions. The result is

$$\ell_1 D^\alpha w = k^2 D^\alpha\left[p(x)u_2\right] - k^2 D^\alpha(v_1 w) + k^2 v_1 D^\alpha w \quad \text{in } \mathbf{R}^3. \tag{6}$$

It is proved in [1] that $w = 0$ outside $\mathcal{D}$. (This follows from the fact that w solves the problem $\ell_0 w = 0$ in $\mathbf{R}^3_+$, $w = 0$ on P, w satisfies the radiation condition. This problem has at most one solution, so $w = 0$ in $\mathbf{R}^3_+$, and, by the unique continuation property for solutions to Helmholtz's equation, $w = 0$ outside $\mathcal{D}$.) Multiply (6) by an arbitrary solution ϕ to the equation $\ell_1\phi := \ell_{v_1}\phi = 0$ in $\mathcal{D}_1 \supset \mathcal{D}$, where $\mathcal{D}_1$ is the domain to which v_1 admits an analytic continuation, integrate over $\mathcal{D}_1$ and then by parts.

Since $D^\beta w = 0$ outside $\mathcal{D}$ for all β, the boundary terms vanish, and since $\ell_1\phi = 0$ in $\mathcal{D}_1$, the volume integral vanishes on the left hand side. One obtains (4).

Step 2. Choose ϕ to be the solution to the equation

$$\ell_1\phi = -\delta(x - z) \quad \text{in } \mathbf{R}^3, \quad z \notin \mathcal{D}, \quad \phi \quad \text{satisfies (2)}. \tag{7}$$

In (7) we assume that v_1 is extended analytically to $\mathcal{D}_1 \supset \mathcal{D}$, and then set to be zero outside of $\mathcal{D}_1$. This allows one to consider the integral in (4) as the result of action of the distribution in brackets with support in $\mathcal{D}$ on a $C^\infty(\mathcal{D})$ function $\phi(x)$. Actually $\phi(x)$ is analytic in $\mathcal{D}$. It is proved in [1] that

$$u_2(x,k) = (4\pi|x|)^{-1} + ik(4\pi)^{-1} + 0(k^2), \quad D^m w = 0(k^2), \quad m = 0, 1, 2 \dots, \quad \text{as } k \to 0 \tag{8}$$

$$\phi(x,k) = (4\pi|x - z|)^{-1} + ik(4\pi)^{-1} + 0(k^2) \quad \text{as } k \to 0. \tag{9}$$

Substitute (8) and (9) in (4), use Leibniz's formula, and equate coefficients in front of k^0 and k to get

$$0 = \int_{\mathcal{D}} |x - z|^{-1} \sum_{m+\ell=|\alpha|} \frac{|\alpha|!}{m!\ell!} D^m p D^\ell(|x|^{-1}) dx, \quad \forall z \notin \mathcal{D} \tag{10}$$

and

$$0 = \int_{\mathcal{D}} \left\{ \sum_{m+\ell=|\alpha|} \frac{|\alpha|!}{m!\ell!} D^m p D^\ell\left(|x|^{-1}\right) + \frac{D^\alpha p}{|x - z|} \right\} dx, \quad \forall z \notin \mathcal{D}. \tag{11}$$

Note that the integrals in (10), (11) are taken over $\mathcal{D}$ (rather than $\mathcal{D}_1$) since $p(x)$ vanishes outside $\mathcal{D}$. Take $|z| \to \infty$ in (9) to get

$$0 = \int_{\mathcal{D}} \sum_{m+\ell=|\alpha|} \frac{|\alpha|!}{m!\ell!} D^m p D^\ell \left(|x|^{-1}\right) dx. \tag{12}$$

From (12) and (11) it follows that

$$\int_{\mathcal{D}} |x-z|^{-1} D^\alpha p(x) dx = 0 \quad \forall z \notin \mathcal{D}. \tag{13}$$

It is proved in [1] that (13) implies

$$\int_{\mathcal{D}} h(x) D^\alpha p(x) dx = 0 \quad \forall h \in \mathcal{H} := \left\{ h : \nabla^2 h = 0 \quad \text{in } \mathcal{D} \right\}. \tag{14}$$

As in [1], equation (14) implies (5). From (5) and (3) it follows that $p(x) = 0$ in $\mathcal{D}$, so $v_1(x) = v_2(x)$ in $\mathcal{D}$. Theorem 1 is proved.

For convenience of the reader let us repeat the argument from [1] which shows that (14) implies (5). Take $|\alpha| = 0$ in (14). Then $\int_{\mathcal{D}} h(x)p(x)dx = 0 \quad \forall h \in \mathcal{H}$. If $h \in \mathcal{H}$ then $\partial h/\partial x_j \in \mathcal{H}$. Thus, using (14) with $|\alpha| = 1$ one obtains

$$0 = \int_{\mathcal{D}} p \partial h/\partial x_j dx = \int_\Gamma p h N_j ds - \int_{\mathcal{D}} h \partial p/\partial x_j dx = \int_\Gamma p h N_j ds. \tag{15}$$

Choose h such that $h = \overline{p} N_j$ on Γ, where the bar stands for complex conjugate. This is possible since the Dirichlet problem for the Laplace equation in $\mathcal{D}$ is (uniquely) solvable. Then (15) yields

$$0 = \int_\Gamma |p|^2 N_j^2 ds \quad j = 1, 2, 3. \tag{16}$$

It follows from (16) that $p = 0$ on Γ so (5) holds with $m = 0$. Similar argument shows that (14) implies (5) for all $m = 0, 1, 2, \ldots$.

References

1. A.G. Ramm, Uniqueness result for inverse problem of geophysics I, *Inverse Problems* 6 (1990).
2. A.G. Ramm, Uniqueness theorems for geophysical problems with incomplete surface data, *Appl. Math. Lett.* **3** (1990).
3. A.G. Ramm, *Scattering by obstacles*, Reidel, Dordrecht, (1986).
4. A.G. Ramm, Multidimensional inverse problems and completeness of the products of solutions to PDE, *J. Math. Anal. Appl.* **134,139**, 211–253, 302 (1988,1989).
5. A.G. Ramm, Numerical method for solving $3D$ inverse scattering problems, *Appl. Math. Lett.* **1**, 381–384 (1988).
6. A.G. Ramm, Numerical recovery of the $3D$ potential from fixed energy incomplete scattering data, *Appl. Math. Lett.* **2**, 101–104 (1989).

Mathematics Department, Kansas State University, Manhattan, KS 66506, USA